Modeling $f(R)$ gravity in terms of mass dilation rate

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We review the conformal equivalence in describing the background expansion of the universe by $f(R)$ gravity both in the Jordan frame and the Einstein frame. In the Jordan frame, we present the general analytic expression for $f(R)$ models that have the same expansion history as the ΛCDM model. This analytic form can provide further insights on how cosmology can be used to test the $f(R)$ gravity at the largest scales. Moreover we present a systematic and self-consistent way to construct the viable $f(R)$ model in Jordan frame using the mass dilation rate function from the Einstein frame through the conformal transformation. In addition, we extend our study to the linear perturbation theories and we further exhibit the equivalence of the $f(R)$ gravity presented in the Jordan frame and Einstein frame in the perturbed space-time. We argue that this equivalence has solid physics root.

I. INTRODUCTION

There has been conclusive evidence indicating that our universe is experiencing an accelerated expansion. This acceleration is believed driven by a so called dark energy (DE) in the framework of Einstein's general relativity. The simplest explanation of such DE is the cosmological constant. But it suffers serious problems such that its value is far below the prediction of any sensible quantum field theories and it inevitably sustains the coincidence problem with the mysterious same order energy density as the matter field today.

There exists an alternative way to explain the acceleration of the universe expansion by modifying the Einstein's gravity. One simplest attempt is called $f(R)$ gravity, in which the scalar curvature in the Lagrange density of Einstein's gravity is replaced by an arbitrary function of R [1]. However it is quite non-trivial to construct the viable $f(R)$ model which can satisfy both cosmological and local gravity constraints [2–10]. Only a few viable $f(R)$ models with simple analytic forms have been found [10–13]. However, if we do not limit to these $f(R)$ models with explicit expressions, in a more general case, $f(R)$ gravity in cosmology can leave enough freedoms to accommodate any desired expansion histories of the universe [14] [15]. A general phenomenological $f(R)$ cosmological model were formulated numerically by specifying the effective DE equation of state w and the present value of the parameter $B \equiv \frac{d \ln F}{d \ln H}$ [15, 16].

On the other hand, it is always possible to transform the theory of $f(R)$ gravity presented in the Jordan frame to the Einstein frame by using the conformal transformation [17]. In the Einstein frame, the model contains the coupling between the matter and an additional scalar field [18]. If the scalar field plays a role of DE in the universe, DE would interact with DM. The phenomenological DE and DM interacting model has been studied extensively in literatures, see [19–25] and references therein. More interestingly, it was found that a general $f(R)$ gravity can be consistently constructed in a covariant form in terms of the mass dilation rate function in the Einstein frame[26]. This mass dilation rate function marks the coupling strength between DE and DM. The condition for the $f(R)$ cosmology to avoid the instability at high-curvature region and be consistent with CMB observation require the energy flow from DE to DM in the framework of the interacting model [26]. This was found helpful to alleviate the coincidence problem[19, 23].

In this paper, we will further examine the equivalence of the $f(R)$ gravity presented in the Jordan frame and the Einstein frame to describe the cosmology. In the background Jordan frame, we will derive the analytic $f(R)$ functions to give the same expansion history of the universe as the ΛCDM model. This can provide further insights on how cosmology can be used to test the $f(R)$ gravity at the largest scales. In addition to modeling $f(R)$ gravity in Jordan frame using the effective DE equation of state, we will present a systematic and self-consistent way to construct the $f(R)$ model in terms of the mass dilation rate function in Einstein frame. Moreover we will develop linear perturbation theories to further exhibit the equivalence of the $f(R)$ gravity in the Jordan frame and the Einstein frame.

This paper is orgnized as follows: In section II, we will discuss the $f(R)$ gravity in the background cosmology. In the Jordan frame, we will derive the general analytical solutions for the $f(R)$ model which has the same expansion history as the ΛCDM cosmology. In the Einstein frame, we will argue that once the mass dilation is specified, the $f(R)$ gravity can be constructed accordingly. In section III, we will extend our discussion to the perturbed space-time. We will show that perturbation theories in Jordan frame and Einstein frame can be consistently connected by conformal transformations. In section IV, we will construct a viable $f(R)$ cosmological model in terms of mass dilation rate specified in the Einstein frame and discuss the subhorizon approximation of the constructed $f(R)$ model. Our conclusion and discussion will be presented in the last section.

II. BACKGROUND DYNAMICS IN $f(R)$ COSMOLOGY

We start with the 4-dimensional action in $f(R)$ gravity [27]

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} f(R) + \int d^4x \mathcal{L}^{(m)} \quad , \qquad (1)$$

where $\kappa^2 = 8\pi G$. The field equation can be obtained by varying the above action with respect to $g_{\mu\nu}$, which leads to

$$FR_{\mu\nu} - \frac{1}{2}fg_{\mu\nu} - \nabla_\mu\nabla_\nu F + g_{\mu\nu}\Box F = \kappa^2 T_{\mu\nu}^{(m)} \quad , \quad (2)$$

where $F = \frac{\partial f}{\partial R}$ and

$$\Box F = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\mu F) \quad . \qquad (3)$$

If we define the left hand side of Eq. (2) as a tensor $\Sigma_{\mu\nu}$

$$\Sigma_{\mu\nu} = FR_{\mu\nu} - \frac{1}{2}fg_{\mu\nu} - \nabla_\mu\nabla_\nu F + g_{\mu\nu}\Box F \quad , \qquad (4)$$

we can recast the modified Einstein's equation into a similar form to the standard Einstein equation

$$\Sigma_{\mu\nu} = \kappa^2 T_{\mu\nu}^{(m)} \quad . \qquad (5)$$

The trace of the above equation gives rise to a scalar equation

$$f = \frac{3}{2}\Box F + \frac{1}{2}FR - \frac{\kappa^2}{2}T^{(m)} \quad . \qquad (6)$$

Taking the derivative of f and further noting that

$$\nabla_\mu f = F\nabla_\mu R \quad , \qquad (7)$$

we obtain

$$\nabla_\mu R = -\frac{\kappa^2}{F}\nabla_\mu T^{(m)} + \frac{3}{F}\nabla_\mu\Box F + R\nabla_\mu\ln F \quad . \qquad (8)$$

We can see that F is a free scalar, which can be used to characterize the $f(R)$ gravity.

A. The Jordan frame

In the Jordan frame, we consider the universe which is described by the flat Friedmann-Robertson-Walker(FRW) metric

$$ds^2 = -dt^2 + a^2 dx^2. \qquad (9)$$

From Eq.(5), we can obtain the equation governing the dynamics in the background spacetime

$$H^2 = \frac{FR - f}{6F} - H\frac{\dot{F}}{F} + \frac{\kappa^2}{3F}\rho_m \quad . \qquad (10)$$

In the Jordan frame, the matter field is conserved, which satisfies

$$\dot{\rho}_m + 3H\rho_m = 0 \quad , \qquad (11)$$

where dot denotes the derivative with respect to the cosmic time t.

From Eq.(10), f can be presented as

$$f = FR - 6FH^2 - 6H^2\frac{dF}{dx} + 2\kappa^2\rho_m \quad . \qquad (12)$$

where $x = \ln a$.

Taking the derivative of the above equation and further noting that $\frac{df}{dx} = F\frac{dR}{dx}$, we obtain

$$\frac{d^2}{dx^2}F + (\frac{1}{2}\frac{d\ln E}{dx} - 1)\frac{dF}{dx} + \frac{d\ln E}{dx}F + \frac{3\Omega_m^0 e^{-3x}}{E} = 0 \quad , \qquad (13)$$

where

$$E \equiv \frac{H^2}{H_0^2} \quad ,$$

$$\Omega_m^0 \equiv \frac{\kappa^2\rho_m^0}{3H_0^2} \quad , \qquad (14)$$

$$R = 3(\frac{dE}{dx} + 4E) \quad . \qquad (15)$$

R is in the unit of H_0^2 and $\kappa^2 = 1$.

For illustrative purposes, we take an expansion history in the Jordan frame that matches a DE model with equation of state w,

$$E = \Omega_m^0 e^{-3x} + (1 - \Omega_m^0)e^{-3\int_0^x(1+w)dx} \quad . \qquad (16)$$

In [15] a family of numerical solutions of $f(R)$ models that match the representative expansion histories of DE model was obtained. Here we find that for the $f(R)$ model to mimic the same expansion history as that of the ΛCDM cosmology in the background $E = \Omega_m^0 e^{-3x} + (1 - \Omega_m^0)$, we can analytically solve Eq.(13) and obtain the general analytical solution

$$F(x) - 1 =$$
$$C(e^{3x})^{p_-}(1 - \Omega_m^0)^{p_-}(\Omega_m^0)^{-p_-} {}_2F_1[q_-, p_-, r_-, z]$$
$$+ D(e^{3x})^{p_+}(1 - \Omega_m^0)^{p_+}(\Omega_m^0)^{-p_+} {}_2F_1[q_+, p_+, r_+, z] \quad , \qquad (17)$$

where ${}_2F_1(a, b; c; z)$ is the hypergeometric function, which is defined as

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!} \quad ,$$
$$(a)_n = a(a+1)\cdots(a+n-1) \quad .$$

C and D are coefficients which will be determined by

boundary conditions. The indexes in the solution read,

$$p_+ = \frac{5+\sqrt{73}}{12} \quad ,$$

$$p_- = \frac{5-\sqrt{73}}{12} \quad ,$$

$$q_+ = \frac{1+\sqrt{73}}{12} \quad ,$$

$$q_- = \frac{1-\sqrt{73}}{12} \quad ,$$

$$r_+ = 1+\frac{\sqrt{73}}{6} \quad ,$$

$$r_- = 1-\frac{\sqrt{73}}{6} \quad . \tag{18}$$

and

$$z = e^{3x}(1-\frac{1}{\Omega_m^0}) \quad ,$$

$$e^{3x} = \frac{3\Omega_m^0}{R-12(1-\Omega_m^0)} \quad . \tag{19}$$

Since the index p_- is negative, the fist term of Eq.(17) will diverge when x goes to the minus infinity, which is not acceptable in physics. Besides, a viable $f(R)$ model should be in a form of "chameleon" type [28], which means that in the high curvature region it should go back to the standard Einstein gravity to pass the local Solar-System tests. The boundary condition for the scalar field $F(x)$ thus requires

$$\lim_{x\to-\infty} F(x) = 1 \quad , \tag{20}$$

and we need to set $C = 0$. The solution turns out to be

$$F(x) = 1+D(e^{3x})^{p_+}(1-\Omega_m^0)^{p_+}(\Omega_m^0)^{-p_+}{}_2F_1[q_+,p_+,r_+,z] \quad . \tag{21}$$

D above is a free parameter which characterizes the different $f(R)$ models which have the same expansion history as that of the ΛCDM model.

With the solution of $F(x)$, we can easily figure out the analytical expression for $f(R)$ by doing the integration

$$f(R) = f(R(z)) = \int F(z)\frac{\partial R}{\partial z}dz \quad . \tag{22}$$

We obtain

$$f(R) = [L_\Lambda - 6(1-\Omega_m^0)](1-\epsilon) + \text{Constant} \quad , \tag{23}$$

where

$$\epsilon = D\frac{\Gamma_E[s_+]}{\Gamma_E[p_+]}(1-\Omega_m^0)^{p_+}(\Omega_m^0)^{-p_+}(e^{3x})^{p_+}{}_2F_1[q_+,s_+,r_+,z] \quad ,$$

and

$$L_\Lambda = R - 2\Lambda = R - 6(1-\Omega_m^0) \quad ,$$
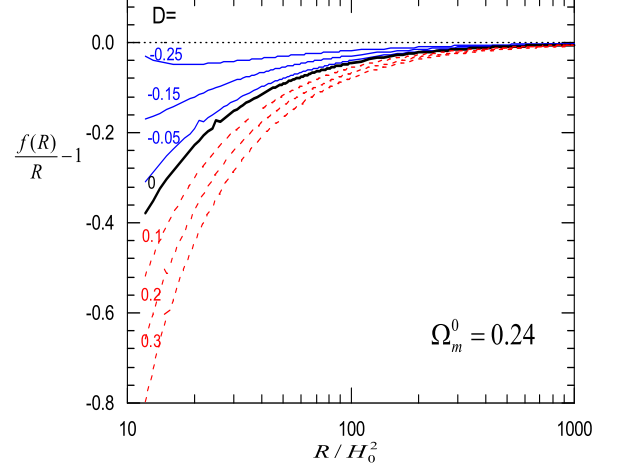
$$s_+ = \frac{-7+\sqrt{73}}{12} \quad . \tag{24}$$

FIG. 1: $f(R)$ models that have the same expansion history as the ΛCDM cosmology.

Γ_E is the complete Euler Gamma function, which is defined as

$$\Gamma_E(z) = \int_0^\infty t^{z-1}e^{-t}dt \quad . \tag{25}$$

When $D = 0$, Eq.(23) should be consistent with Eq.(12) and should go back to the ΛCDM model. This puts the constant in Eq.(23) into $6(1-\Omega_m^0)$. Thus finally we have the analytic $f(R)$ form

$$f(R) = [L_\Lambda - 6(1-\Omega_m^0)](1-\epsilon) + 6(1-\Omega_m^0) \quad . \tag{26}$$

Eq.(26) is a general analytical expression for $f(R)$ models that can have the same background expansion history as that of the ΛCDM cosmology. The solution has only two free dimensionless parameters, namely, Ω_m^0 and D. In order to avoid the short-timescale instability at high curvature [15], it requires $D < 0$. Moreover for a viable $f(R)$ model, it also requires $F > 0, f_{RR} > 0$ [18]. In Fig 1, we illustrate the $f(R)$ models obtained from Eq.(26) and the blues lines indicate the viable $f(R)$ models which satisfy the condition mentioned above.

If we do not start from the concrete model describing the expansion history of the universe, there is a lot of freedom for choosing the scalar field $F(x)$ except the boundary condition Eq.(20). The $f(R)$ gravity can be constructed only when the scalar field $F(x)$ can be reasonably chosen based on physical motivations. However, in the Jordan frame, there is no clear physical motivation to provide a particular form for the field $F(x)$. We need to turn to the Einstein frame. In the Einstein frame, $F(x)$ relates to a scalar $\Gamma(x)$ representing the mass dilation rate $\frac{\partial \ln m}{\partial t}$ in gravitational field [13, 26] and $\Gamma(x)$ can be deemed as the coupling strength in the framework

of interacting dark energy models [19, 20, 24, 25]. $F(x)$ can be determined by constructing Γ which, in turn, the $f(R)$ model can be constructed. In the next section we will shift our attention to the Einstein frame.

B. The Einstein frame

The equations of motion in the Einstein frame can be obtained by using conformal transformations

$$\tilde{H} = \frac{1}{\sqrt{F}}(1 + \frac{1}{2}\frac{d\ln F}{dx})H \quad . \tag{27}$$

As shown in [26], after the conformal transformation the expansion history in the Einstein frame is governed by

$$\frac{d\tilde{H}}{d\tilde{x}} = \frac{\tilde{R}}{6\tilde{H}} - 2\tilde{H} \quad , \tag{28}$$

$$\frac{d\tilde{R}}{d\tilde{x}} + 36\Gamma\frac{d\Gamma}{d\tilde{x}} + 72\Gamma^2 = -3\kappa^2\tilde{\rho}_m(\frac{\Gamma}{\tilde{H}} + 1) \quad , \tag{29}$$

$$\frac{d\tilde{\rho}_m}{d\tilde{x}} + 3\tilde{\rho}_m = \frac{\Gamma}{\tilde{H}}\tilde{\rho}_m \quad , \tag{30}$$

where Γ is the coupling strength between the scalar field and matter which can also be explained as the mass dilation rate function [26]. From the relation between F and Γ and expressing F into φ, $\ln F = \kappa\sqrt{\frac{2}{3}}\varphi$, we can rewrite the continuity equation for the scalar field into

$$\frac{d\varphi}{d\tilde{x}} = -\frac{\sqrt{6}\Gamma}{\kappa\tilde{H}}. \tag{31}$$

Once the dilation function is specified, we can explore the expansion history of the universe in the Einstein frame.

Γ has clear physical meaning, which represents the coupling strength of the scalar field to the matter field. Although Γ is a free field, the choice of Γ must be consistent with the viability condition of the $f(R)$ gravity. A viable $f(R)$ model must pass the stringent local test, which requires that when $\tilde{R} \to \infty$, the model should go back to the standard Einstein gravity, so that $\lim_{\tilde{x}\to-\infty}\Gamma = 0$.

Eqs.(28)-(30) governing the expansion history are not stable when $\Gamma < 0$ at high curvature region, except that the coupling strength is extremely small, for example $|\Gamma| < 10^{-10}$. This phenomena is consistent with that discussed in [29]. Thus a viable $f(R)$ model requires that Γ is positive and drops to sufficiently small values in the early time of the Universe.

Γ also is the dilation rate of the mass of test particles in gravitational fields. To satisfy the weak equivalent principle, Γ should be independent of the species of matter [30]. One natural choice of the Γ form is to consider it as a geometrical quantity, a function of $\tilde{R}$. There is quite a wide range of the choices for Γ which can satisfy the above conditions. In this work, we only present one of the simplest choices, where we describe Γ as a function

of the scalar Ricci curvature $\tilde{R}$ in the Einstein frame as

$$\Gamma = \frac{\alpha}{\tilde{R} + \beta} \quad , \tag{32}$$

where α and β are constants. Eq.(32) is just one kind of possible phenomenological descriptions since we are lack of the knowledge on the nature of the dilation rate.

Once we specify the dilation rate Γ, we can get the $f(R)$ gravity in the Jordan frame by using the conformal transformation from the Einstein frame to the Jordan frame.

III. PERTURBATION THEORY IN $f(R)$ COSMOLOGY

In this section, we will extend our discussion to the perturbed space-time. We will first review the cosmological perturbation theory in $f(R)$ gravity in the Jordan frame. We will show that perturbations can be consistently connected between the Jordan frame and the Einstein frame.

A. The Jordan frame

We focus on the scalar perturbation. The perturbed line element in Fourier space reads,

$$ds^2 = a^2[-(1 + 2\psi Y^{(s)})d\tau^2 + 2BY_i^{(s)}d\tau dx^i$$
$$+ (1 + 2\phi Y^{(s)})\delta_{ij}dx^i dx^j + EY_{ij}^{(s)}dx^i dx^j] \quad , \tag{33}$$

where ψ, B, ϕ, E are scalar fields which indicate the perturbations. $Y^{(s)}$ is the scalar harmonic function. The perturbed form of the modified Einstein equations Eq.(5)

$$\delta\Sigma_\nu{}^\mu = \kappa^2\delta T_{(m)\nu}{}^\mu$$

can be obtained by writing

$$\delta\Sigma_\nu{}^\mu = \delta g^{\mu\sigma}\Sigma_{\nu\sigma} + g^{\mu\sigma}\delta\Sigma_{\nu\sigma} \quad , \tag{34}$$

where

$$\delta\Sigma_{\mu\nu} = \delta F R_{\mu\nu} + F\delta R_{\mu\nu} - \frac{1}{2}Fg_{\mu\nu}\delta R - \frac{1}{2}f\delta g_{\mu\nu}$$
$$- \delta\nabla_\mu\nabla_\nu F + \delta g_{\mu\nu}\Box F + g_{\mu\nu}\delta\Box F \quad , \tag{35}$$

and expressing

$$\delta T_{(m)0}{}^0 Y^{(s)} = -\delta\rho_m Y^{(s)} \quad ,$$
$$\delta T_{(m)i}{}^0 Y_i^{(s)} = (p_m + \rho_m)(v_m + B)Y_i^{(s)} \quad ,$$
$$\delta T_{(m)0}{}^i Y^{(s)i} = -(p_m + \rho_m)v_m Y^{(s)i} \quad ,$$
$$\delta T_{(m)j}{}^i Y_j^{(s)i} = (\delta p_m\delta_j^i + \Pi_{mj}{}^i)Y_j^{(s)i} \quad . \tag{36}$$

Inserting the line element Eq.(33) into the above equations, we can get the perturbation form of the modified Einstein equation.

From $\delta\Sigma_0{}^0 = \kappa^2 \delta T_{(m)0}{}^0$ we obtain,

$$
\begin{aligned}
-\frac{\kappa^2}{2}\delta\rho_m a^2 \;=\; & F[-k^2\phi + 3\mathcal{H}(\mathcal{H}\psi - \phi') + k\mathcal{H}B - \frac{1}{6}k^2 E] \\
& + F'(\frac{k}{2}B + 3\mathcal{H}\psi - \frac{3}{2}\phi') - \frac{3}{2}\mathcal{H}^2\delta F \\
& + \frac{3}{2}\frac{a''}{a}\delta F - \frac{3}{2}\mathcal{H}\delta F' - \frac{k^2}{2}\delta F \quad ,
\end{aligned}
\tag{37}
$$

where the prime denotes the derivative with respect to the conformal time τ.

The trace part $\delta\Sigma_i{}^i = \kappa^2 \delta T_{(m)i}{}^i$ gives

$$
\begin{aligned}
\kappa^2 a^2 \delta p_m \;=\; & F[-\frac{2}{3}k^2\psi - \frac{2}{3}k^2\phi - \frac{1}{9}k^2 E + 4\frac{a''}{a}\psi + \frac{2}{3}kB' \\
& + \frac{4}{3}k\mathcal{H}B - 2\mathcal{H}^2\psi + 2\mathcal{H}\psi' - 4\mathcal{H}\phi' - 2\phi''] \\
& + F'[\frac{2}{3}kB + \psi' + 2\mathcal{H}\psi - 2\phi'] \\
& + \delta F[\mathcal{H}^2 + \frac{a''}{a} - \frac{2}{3}k^2] - \delta F'' - \delta F'\mathcal{H} + 2\psi F'' \quad ,
\end{aligned}
\tag{38}
$$

and the anisotropic part $\delta\Sigma_i{}^j = \kappa^2 \delta T_{(m)i}{}^j$ $(i \neq j)$ gives

$$
\begin{aligned}
\kappa^2 a^2 p_m \Pi_m \;=\; & F[-k^2\psi - k^2\phi + 2k\mathcal{H}B + \mathcal{H}E' - \frac{1}{6}k^2 E \\
& + \frac{1}{2}E'' + kB'] + F'(\frac{1}{2}E' + kB) - k^2\delta F.
\end{aligned}
\tag{39}
$$

The last part $\delta\Sigma_0{}^i = \kappa^2 \delta T_{(m)0}{}^i$ leads to

$$
\begin{aligned}
-\frac{1}{2}\kappa^2 a^2(\rho_m + p_m)v_m \;=\; & F[k\phi' - k\mathcal{H}\psi + 2\mathcal{H}^2 B - \frac{a''}{a}B \\
& + \frac{1}{6}kE'] + F'[\mathcal{H}B - \frac{1}{2}k\psi] \\
& + \frac{1}{2}k\delta F - \frac{1}{2}k\mathcal{H}\delta F - \frac{1}{2}BF''.
\end{aligned}
\tag{40}
$$

The above perturbation equations are covariant. This can be easily seen by investigating the infinitesimal coordinate transformation,

$$
\begin{aligned}
\hat{x^\mu} &= x^\mu + \delta x^\mu \quad , \\
\delta x^0 &= \xi^0 Y^{(s)} \quad , \\
\delta x^i &= \beta Y^{(s)i} \quad .
\end{aligned}
$$

We use the symbol with "hat" to indicate the quantities after coordinate transformation. Under the infinitesimal transformation, we can show that the perturbed quantities in the line element Eq. (33) behave as [25]

$$
\begin{aligned}
\hat{\psi}Y^{(s)} &= (\psi - \xi^{0'} - \mathcal{H}\xi^0)Y^{(s)} \quad , \\
\hat{\phi}Y^{(s)} &= (\phi - \frac{1}{3}k\beta - \mathcal{H}\xi^0)Y^{(s)} \quad , \\
\hat{B}Y_i^{(s)} &= (B - k\xi^0 - \beta')Y_i^{(s)} \quad , \\
\hat{E}Y_{ij}^{(s)} &= (E + 2k\beta)Y_{ij}^{(s)} \quad .
\end{aligned}
\tag{41}
$$

After inserting Eq. (41) into Eqs. (37)-(40), we find that Eqs. (37)-(40) keep the same forms under the infinitesimal transformation, which shows that Eqs. (37)-(40) are covariant. Furthermore, if we take $F \to 1, \delta F \to 0$, Eqs. (37)-(40) can go back to the standard cosmological perturbation equations in the Einstein gravity as presented in [31].

After we get the perturbed form of the modified Einstein equations, we turn to derive the equation of motion for matter field in the Jordan frame. From

$$
\nabla^\nu T_{\mu\nu}^{(m)} = 0
\tag{42}
$$

we can obtain its perturbed form $\delta\nabla^\nu T_{\mu\nu}^{(m)} = 0$. Looking at the zero-th component, we have

$$
\begin{aligned}
& \delta_m' + 3\mathcal{H}(\frac{\delta p_m}{\delta\rho_m} - w_m)\delta_m \\
& = -(1 + w_m)kv_m - 3(1 + w_m)\phi' \quad .
\end{aligned}
\tag{43}
$$

The i-th component gives

$$
\begin{aligned}
& [(\rho_m + p_m)(B + v_m)]' + 4\mathcal{H}(p_m + \rho_m)(B + v_m) \\
& = (p_m + \rho_m)k\psi + k\delta p_m - \frac{2}{3}kp_m\Pi_m \quad .
\end{aligned}
\tag{44}
$$

Similarly, it can be shown that the above equations are covariant under the infinitesimal coordinate transformation.

The freedom of choosing coordinates in general relativity will lead to ambiguity in defining a real density perturbation from the coordinate transformation. The systematic way to get rid of the ambiguity is to fix the gauge, which has been well discussed in [31]. Here we will argue that the gauge conditions in the Jordan frame are not always the same as the conditions in the Einstein frame after the conformal transformation.

We only focus our discussion on the most commonly used gauges, the Newtonian gauge and the Synchronous gauge. The Newtonian gauge is defined by setting $B = E = 0$ and these conditions completely fix the gauge

$$
\begin{aligned}
\xi^0 &= -\frac{\hat{B}}{k} - \frac{\hat{E}'}{2k^2} \quad , \\
\beta &= \frac{\hat{E}}{2k} \quad .
\end{aligned}
\tag{45}
$$

The perturbations in this gauge can be shown as gauge invariant. For instance, the matter density perturbation can be presented as

$$
\delta^{(N)} = \hat{\delta} - \frac{\rho_m'}{\rho_m}(\frac{\hat{B}}{k} + \frac{\hat{E}'}{2k^2}) \quad ,
\tag{46}
$$

where $\hat{\delta}$ is a general perturbation. We can show that $\delta^{(N)}$ is invariant under the coordinate transformation Eq.(41).

However, in contrast to the Newtonian gauge, the Synchronous gauge is not completely fixed because the gauge

condition $\psi = B = 0$ only confines the gauge up to two arbitrary constants C_1, C_2,

$$\xi^0 = \frac{C_1}{a} \quad ,$$
$$\beta = -kC_1 \int \frac{d\tau}{a} + C_2 \quad . \tag{47}$$

In order to fix the Synchronous gauge, additional conditions are called for. Usually, C_2 is fixed by specifying the initial condition for the curvature perturbation in the early time of the universe and C_1 can be fixed by setting the peculiar velocity of DM to be zero, $v_m = 0$, which is equivalent to say that the gauge is at rest with respect to the cold DM and the observer is comoving with cold DM. After fixing C_1, C_2, the Synchronous gauge can be completely fixed.

As long as the Synchronous gauge is completely fixed, it will be equivalent to the Newtonian gauge. For instances, the density perturbations in different gauges are related by

$$\delta^{(N)} = \delta^{(S)} + \frac{\rho'_m}{\rho_m} \frac{v^N - v^S}{k} \quad , \tag{48}$$

where $v^N = \hat{v} - \frac{\hat{E}'}{2k}$, $v^S = 0$.

We will show later that the gauge condition for the Newtonian gauge will be the same both in the Jordan frame and the Einstein frame. However we will see that the gauge condition for the Synchronous gauge will change when we transform the Jordan frame to the Einstein frame by using the conformal transformation.

Since the Newtonian gauge has this advantage, we will work in the Newtonian gauge hereafter. We use the Bardeen potentials [32] $\Phi = \phi$, $\Psi = \psi$ to represent the space time perturbations. Since we are only interested in the DM dominated period in the $f(R)$ cosmology, we set $p_m = 0$ and $\delta p_m = 0$. The perturbed Einstein equations can be reduced to

$$-\frac{\kappa^2}{2}\delta\rho_m a^2 = F[-k^2\Phi + 3\mathcal{H}(\mathcal{H}\Psi - \Phi')]$$
$$+ F'(3\mathcal{H}\Psi - \frac{3}{2}\Phi') - \frac{3}{2}\mathcal{H}^2\delta F$$
$$+ \frac{3}{2}\frac{a''}{a}\delta F - \frac{3}{2}\mathcal{H}\delta F' - \frac{k^2}{2}\delta F \quad , \tag{49}$$

$$\delta F'' = -\mathcal{H}\delta F' + F[-\frac{2}{3}k^2\Psi - \frac{2}{3}k^2\Phi + 4\frac{a''}{a}\Psi$$
$$- 2\mathcal{H}^2\Psi + 2\mathcal{H}\Psi' - 4\mathcal{H}\Phi' - 2\Phi''] + 2\Psi F''$$
$$+ F'[\Psi' + 2\mathcal{H}\Psi - 2\Phi']$$
$$+ \delta F[\mathcal{H}^2 + \frac{a''}{a} - \frac{2}{3}k^2] \quad , \tag{50}$$

$$\frac{\delta F}{F} = -\Psi - \Phi \quad , \tag{51}$$

$$-\frac{1}{2}\kappa^2 a^2 \rho_m v_m = F[k\Phi' - k\mathcal{H}\Psi] - \frac{1}{2}kF'\Psi$$
$$+ \frac{1}{2}k\delta F' - \frac{1}{2}k\mathcal{H}\delta F \quad . \tag{52}$$

F is a free scalar field which characterizes the deviation from the Einstein gravity. The perturbation δF can be obtained by

$$\delta F = \frac{\partial F}{\partial R}\delta R \quad , \tag{53}$$

where

$$a^2\delta R = \mathcal{H}(18\Phi' - 6\Psi') + 4k^2\Phi + 2k^2\Psi - 12\Psi\frac{a''}{a} + 6\Phi'' \quad . \tag{54}$$

Using Eq.(54) and Eq.(49) to eliminate Φ'' and $k^2\Phi$ in Eq.(50), we obtain

$$\delta F'' + 2\mathcal{H}\delta F' + a^2(\frac{k^2}{a^2} + M^2)\delta F$$
$$= \frac{\kappa^2}{3}a^2\delta\rho_m + F'(4\mathcal{H}\Psi - 3\Phi' + \Psi') + 2\Psi F'' \quad , \tag{55}$$

where

$$M^2 = \frac{1}{3}(\frac{F}{f_{RR}} - R) \quad , \tag{56}$$

and

$$R = \frac{6a''}{a^3} \quad , \quad f_{RR} = \frac{\partial F}{\partial R} \quad . \tag{57}$$

Eq.(55) is consistent with Eq.(8.92) in [18], if we express it in the cosmic time t.

¿From the equations of motion Eqs.(43) (44), we obtain

$$\delta'_m = -kv_m - 3\Phi' \quad , \tag{58}$$
$$v'_m + \mathcal{H}v_m = k\Psi \quad . \tag{59}$$

Combining Eqs.(58, 59), we have the second order differential equation for the evolution of the matter perturbation

$$\delta''_m + \mathcal{H}\delta'_m + k^2\Psi = -3\mathcal{H}\Phi' - 3\Phi'' \quad . \tag{60}$$

B. The Einstein frame

We will first derive the perturbed form of the equation of motion for matter fields Eq.(42) under the general conformal transformation. Then we will fix the freedom of the general conformal transformation by setting $\Omega^2 = F$ to obtain the perturbation form of the modified Einstein equations. We will show that the equations obtained under conformal transformation from the Jordan frame are consistent with the results derived by doing perturbations directly in the Einstein frame.

In the background, the conformal transformation is defined by re-scaling the line element of the Jordan frame

$$\tilde{ds}^2 = \Omega^2 ds^2 \quad . \tag{61}$$

However, in perturbed spacetime, we need to take account of the perturbation of Ω. And thus the full line element reads

$$
\begin{aligned}
\tilde{ds}^2 &= (\Omega + \delta\Omega)^2 ds^2 \\
&= \tilde{a}^2[-(1 + 2\tilde{\psi})d\tau^2 + 2\tilde{B}d\tau dx^i \\
&\quad + (1 + 2\tilde{\phi})\delta_{ij}dx^i dx^j + \tilde{E}dx^i dx^j] \quad . \quad (62)
\end{aligned}
$$

Since we are only interested in the linear perturbation, we can expand the above expression to the linear order

$$
\begin{aligned}
\psi &= \tilde{\psi} - \delta\ln\Omega \quad , \\
\phi &= \tilde{\phi} - \delta\ln\Omega \quad , \\
B &= \tilde{B} \quad , \\
E &= \tilde{E} \quad , \quad (63)
\end{aligned}
$$

where the symbols with "tilde" indicate the quantities in the Einstein frame.

For the clear discussion, hereafter we will use "tilde" to indicate quantities that are different in the Einstein frame from their values in the Jordan frame. For the quantities without "tilde", they are the same in both frames. We will use "hat" to indicate the coordinate transformation which should not be confused with the conformal transformation. The differential operator d does not depend on the geometric structure g_{ab} on the spacetime manifold, it will not change under the conformal transformation so that the ordinary derivative operator ∂_a and the conformal coordinates $d\tau, dx$ will be the same in both the Einstein frame and the Jordan frame.

Noting

$$
\delta\hat{\ln}\Omega = \delta\ln\Omega - (\ln\Omega)'\xi^0 \quad , \quad (64)
$$

and considering Eqs.(63,41), we can show that, under the infinitesimal coordinate transformation, the perturbed quantities $\tilde{\psi}, \tilde{\phi}$ behave as

$$
\begin{aligned}
\hat{\tilde{\psi}} &= \tilde{\psi} - \xi^{0\prime} - \tilde{\mathcal{H}}\xi^0 \quad , \\
\hat{\tilde{\phi}} &= \tilde{\phi} - \frac{1}{3}k\beta - \tilde{\mathcal{H}}\xi^0 \quad , \quad (65)
\end{aligned}
$$

where

$$
\mathcal{H} = \tilde{\mathcal{H}} - (\ln\Omega)' \quad . \quad (66)
$$

It is clear that in the coordinate transformation $\tilde{\psi}, \tilde{\phi}$ behave in a similar way as in the Jordan frame. This point also holds for other perturbation quantities

$$
\begin{aligned}
\hat{B} &= B - k\xi^0 - \beta' \quad , \\
\hat{E} &= E + 2k\beta \quad .
\end{aligned}
$$

Now we can discuss the gauge conditions in the Einstein frame. In the Newtonian gauge, the gauge conditions $B = E = 0$ fix the gauge as that in the Jordan frame, namely

$$
\begin{aligned}
\xi^0 &= -\frac{\hat{B}}{k} - \frac{\hat{E}'}{2k^2} \quad , \\
\beta &= \frac{\hat{E}}{2k} \quad . \quad (67)
\end{aligned}
$$

Thus the Newtonian gauge conditions keep the same under the conformal transformation.

However, in the Synchronous gauge, after the conformal transformation, the gauge conditions in the Einstein frame read

$$
\begin{aligned}
\tilde{\psi} &= \delta\ln\Omega \quad , \\
B &= 0 \quad , \\
v_m &= 0 \quad . \quad (68)
\end{aligned}
$$

We will find that $v_m = 0$ is a physical choice because it is the solution of the equation of motion in Eq.(75). It is clear that in the Einstein frame the gauge conditions for the Synchronous gauge are not exactly the same as those in the Jordan frame.

Considering the gauge conditions for the Newtonian gauge are the same in both the Einstein frame and the Jordan frame, we will work in Newtonian gauge hereafter.

In the background, the conformal transformation leads the equation of motion of the matter field Eq.(42) in the Einstein frame to the form

$$
\tilde{\nabla}_\mu \tilde{T}^{\mu\nu} = -\tilde{T}\frac{\tilde{\partial}^\nu\Omega}{\Omega} \quad , \quad (69)
$$

where $\tilde{T}^{\mu\nu} \equiv \tilde{g}^{\mu\tau}\tilde{g}^{\nu\sigma}\tilde{T}_{\tau\sigma}$, $T^{\mu\nu} \equiv g^{\mu\tau}g^{\nu\sigma}T_{\tau\sigma}$ and

$$
\tilde{T}_{\mu\nu} = \frac{1}{\Omega^2}T_{\mu\nu} \quad , \tilde{T}^{\mu\nu} = \frac{1}{\Omega^6}T^{\mu\nu} \quad , \tilde{T} = \frac{T}{\Omega^4}. \quad (70)
$$

$-\tilde{T}\frac{\tilde{\partial}^\nu\Omega}{\Omega}$ is the coupling vector. We only consider the evolving universe where the coupling vector is time-like. We focus on the DM dominated phase with $p_m = 0$.

The conservation equation Eq.(69) can be recasted into

$$
\tilde{U}^0\frac{\partial\tilde{\rho}_m}{\partial\tau} + 3\tilde{H}\tilde{\rho}_m = \Gamma\tilde{\rho}_m \quad , \quad (71)
$$

where $\tilde{U}^0 = 1/\tilde{a}$, Γ is the mass dilation rate

$$
\Gamma = -\tilde{U}^0\frac{\partial\ln\Omega}{\partial\tau} = \frac{1}{\tilde{m}}\tilde{U}^0\frac{\partial\tilde{m}}{\partial\tau}. \quad (72)
$$

The perturbed equation of Eq.(71) reads

$$
\tilde{\delta}'_m = -kv_m - 3\tilde{\Phi}' + \tilde{a}\Gamma\tilde{\Psi} + \tilde{a}\delta\Gamma \quad , \quad (73)
$$

where we have used

$$
\begin{aligned}
3\delta\tilde{H} &= \frac{1}{\tilde{a}}kv_m + \frac{3}{\tilde{a}}(\tilde{\Phi}' - \tilde{\mathcal{H}}\tilde{\Psi}) \quad , \\
\delta\Gamma &= \tilde{\Psi}\tilde{U}^0(\ln\Omega)' - \tilde{U}^0(\delta\ln\Omega)' \quad (74)
\end{aligned}
$$

and $\delta\tilde{U}^0 = -\tilde{\psi}\tilde{U}^0$.

In the Einstein frame, the perturbation equation for the peculiar velocity has the form

$$
v'_m + \tilde{\mathcal{H}}v_m = \tilde{a}\Gamma(\tilde{v}_t - v_m) + k\tilde{\Psi} \quad , \quad (75)
$$

where

$$
\tilde{v}_t = k\frac{\delta\ln\Omega}{(\ln\Omega)'} \quad . \quad (76)
$$

Eqs.(73, 75) are consistent with the results in [23] and [20] by setting the general coupling vector as $\tilde{Q}^\mu = \Gamma \tilde{\rho}_m \tilde{U}_t^\mu$, where $\tilde{U}_t^\mu$ is along the direction of $\tilde{\partial}^\nu \Omega$.

Eqs.(73, 75) can also be consistently obtained from the perturbation equations of Eqs.(58, 59) by doing the conformal transformation

$$a = \frac{\tilde{a}}{\Omega} \quad ,$$
$$v_m = \tilde{v}_m \quad ,$$
$$U^0 = \Omega \tilde{U}^0 \quad ,$$
$$\rho_m = \Omega^4 \tilde{\rho}_m \quad ,$$
$$\delta\rho_m = \delta\Omega^4 \tilde{\rho}_m + \Omega^4 \delta\tilde{\rho}_m \quad , \quad (77)$$
$$\delta_m = \tilde{\delta}_m + 4(\delta\ln\Omega) \quad ,$$
$$\Psi = \tilde{\Psi} - \delta\ln\Omega \quad ,$$
$$\Phi = \tilde{\Phi} - \delta\ln\Omega \quad .$$

Similarly, combining Eqs.(73, 75), we can get the second order differential equation

$$\tilde{\delta}_m'' + (\tilde{\mathcal{H}} + \tilde{a}\Gamma)\tilde{\delta}_m' + k^2\tilde{\psi}$$
$$= (\tilde{\mathcal{H}} + \tilde{a}\Gamma)\tilde{a}\delta\Gamma - k\tilde{a}\Gamma\tilde{v}_t + (\tilde{a}\delta\Gamma)'$$
$$+ (\tilde{\mathcal{H}} + \tilde{a}\Gamma)(\tilde{a}\Gamma\tilde{\psi} - 3\tilde{\phi}') - 3\tilde{\phi}'' + (\tilde{a}\Gamma\tilde{\psi})' \quad . \quad (78)$$

Next, we consider the perturbed Einstein equations in the Einstein frame. We set $\Omega^2 = F$ to fix the freedom of the general conformal transformations. Inserting the conformal transformations Eq.(77) into Eqs.(49)-(52), we obtain

$$-\frac{\kappa^2}{2}(\delta\tilde{\rho}_m + \delta\tilde{\rho}_d)\tilde{a}^2 = -k^2\tilde{\Phi} + 3\tilde{\mathcal{H}}(\tilde{\mathcal{H}}\tilde{\Psi} - \tilde{\Phi}') \quad , \quad (79)$$

$$\kappa^2\tilde{a}^2\delta\tilde{p}_d = -\frac{2}{3}k^2\tilde{\Psi} - \frac{2}{3}k^2\tilde{\Phi} + 4\frac{\tilde{a}''}{\tilde{a}}\tilde{\Psi}$$
$$- 2\tilde{\mathcal{H}}^2\tilde{\Psi} + 2\tilde{\mathcal{H}}\tilde{\Psi}' - 4\tilde{\mathcal{H}}\tilde{\Phi}' - 2\tilde{\Phi}'' \quad ,$$
$$0 = -\tilde{\Psi} - \tilde{\Phi} \quad ,$$
$$k\tilde{\Phi}' - k\tilde{\mathcal{H}}\tilde{\Psi} = -\frac{1}{2}\kappa^2\tilde{a}^2[\tilde{\rho}_m v_m + (\tilde{\rho}_d + \tilde{p}_d)\tilde{v}_d] \quad ,$$

where $\tilde{v}_d$, $\tilde{\rho}_d$ and $\tilde{p}_d$ are defined as

$$\tilde{v}_d = -k\frac{\delta\ln F}{2\tilde{a}\Gamma} \quad ,$$
$$\tilde{\rho}_d = \frac{3\Gamma^2}{\kappa^2} + V \quad ,$$
$$\tilde{p}_d = \frac{3\Gamma^2}{\kappa^2} - V \quad ,$$
$$V = \frac{FR - f}{2\kappa^2 F^2} \quad . \quad (80)$$

The perturbed Einstein equations give rise to the perturbation of δV as

$$\kappa^2\tilde{a}^2\delta V = (\delta\ln F)(9\tilde{a}'\Gamma + 2\tilde{\mathcal{H}}^2 - 3\tilde{a}^2\Gamma^2 - \frac{\tilde{a}''}{\tilde{a}} + 3\tilde{a}\Gamma') \quad . \quad (81)$$

where we have used

$$\delta\Gamma = -\tilde{\Psi}\Gamma - \frac{1}{2\tilde{a}}(\delta\ln F)' \quad . \quad (82)$$

If we directly perturb this V expressed in Eq.(80)

$$\kappa^2\delta V = \frac{1}{2}\frac{\delta F R}{F^2} - \frac{(FR - f)\delta F}{F^3} \quad , \quad (83)$$

after inserting

$$R = 6F(3\frac{\tilde{\mathcal{H}}}{\tilde{a}}\Gamma + \Gamma^2 + \frac{\tilde{a}''}{\tilde{a}^3} + \frac{\Gamma'}{\tilde{a}}) \quad , \quad (84)$$

eliminating f by the Friedmann equation and further employing

$$\kappa^2\tilde{\rho}_m = 4\frac{\tilde{\mathcal{H}}^2}{\tilde{a}^2} - 2\frac{\tilde{a}''}{\tilde{a}^3} - 6\Gamma^2 \quad , \quad (85)$$

we can get the result in consistent with Eq.(81).

δF can be determined by Eq.(55). Inserting Eq.(77), (55) turns out to be

$$\delta F'' = [-k^2 + \frac{2}{3}\kappa^2\tilde{a}^2\tilde{\rho}_m - 2\tilde{a}^2\Gamma^2 + 2\frac{\tilde{a}''}{\tilde{a}}$$
$$+ 4\tilde{a}\Gamma' + 12\tilde{a}'\Gamma - \frac{1}{3}\frac{\tilde{a}^2}{f_{RR}}]\delta F + \frac{1}{3}\kappa^2\tilde{a}^2 F\delta\tilde{\rho}_m$$
$$- (2\tilde{\mathcal{H}} + 4\tilde{a}\Gamma)\delta F' - 2\tilde{a}F\Gamma\tilde{\Psi}' - 4F\tilde{a}\Psi\Gamma'$$
$$- 12F\Psi\tilde{a}'\Gamma + 6F\tilde{a}\tilde{\Phi}'\Gamma \quad , \quad (86)$$

where f_{RR} is given by

$$f_{RR} = \frac{F'}{R'} = \frac{2\tilde{a}F\Gamma}{R'} \quad , \quad (87)$$

and R can be obtained from Eq.(84) in the Einstein frame.

After fixing δF, we can obtain the consistent covariant perturbation for Γ by Eq.(82), since the infinitesimal coordinate transformation for $\delta\Gamma$ presented in Eq.(82) satisfies the rules of the covariant transformation for a scalar field

$$\delta\hat{\Gamma} = \delta\Gamma - \Gamma'\xi^0 \quad . \quad (88)$$

Similarly the perturbation for H, δH is given by its covariant configuration

$$3H = \nabla_\mu U^\mu \quad , \quad (89)$$

and so does $\delta\Gamma$

$$\Gamma = -\tilde{U}^\mu\tilde{\nabla}_\mu\ln\Omega \quad . \quad (90)$$

IV. SUBHORIZION APPROXIMATION

From the above analysis, $f(R)$ model in the Jordan frame is conformally equivalent to the interaction model

in the Einstein frame. This equivalence holds not only in the background dynamics but also in the perturbed spacetime. Therefore, we can construct $f(R)$ models by choosing a peculiar form of the mass dilation rate function Γ in the Einstein frame. One of the simplest choice of the Γ is

$$\Gamma = \frac{\alpha}{\tilde{R} + \beta} \quad .$$

After fixing the mass dilation rate function, we can figure out the evolution of the cosmological model in the Einstein frame and then we can transform it into the Jordan frame by using the conformal transformation. In the lower panel of Fig 2, we present some examples for the effective DE equation of state w for the $f(R)$ cosmology in the Jordan frame constructed by selecting the above mass dilation rate function in the Einstein frame after the conformal transformation. We can see that the effective DE equation of state in the constructed $f(R)$ model presents the phenomena in the vicinity of -1 and with the -1 crossing behavior, which is consistent with present cosmological observations. This shows that with the appropriate choice of the mass dilation function, which refers to the mass, a more fundamental quantity, we can construct the viable $f(R)$ model to produce the observationally allowed effective DE equation of state.

We have also shown that the linear perturbation equations are exactly equivalent at all scales in both the Einstein frame and the Jordan frame. In the usual $f(R)$ model, it was argued that the scale factor described by the $f(R)$ cosmology during the matter phase behaves inconsistently with cosmological observations[9]. It is of interest to examine the $f(R)$ model constructed from the peculiar mass dilation rate function we have chosen and see whether our $f(R)$ model is viable if comparing with observations in the matter dominated phase. We assume that the quantities defined in the Jordan frame are consistent with the actual observational quantities. We will study the subhorizon approximation in the Jordan frame.

Following [18], from Eqs.(49,51), we obtain

$$\Phi \approx -\frac{1}{2}\delta \ln F + \frac{a^2}{k^2}\frac{\kappa^2}{2F}\delta\rho_m \quad ,$$
$$\Psi \approx -\frac{1}{2}\delta \ln F - \frac{a^2}{k^2}\frac{\kappa^2}{2F}\delta\rho_m \quad , \qquad (91)$$

where δF is governed by Eq.(55)

$$\frac{\delta \ln F}{\delta \rho_m} \approx \frac{\kappa^2}{3F}\frac{1}{\frac{k^2}{a^2} + M^2} \quad , \qquad (92)$$

and M^2 is defined by Eq.(56). From Eq.(60), the equation for the growth function $G(z,k) = \frac{d\ln D}{dx}$ in the subhorizon approximation reads

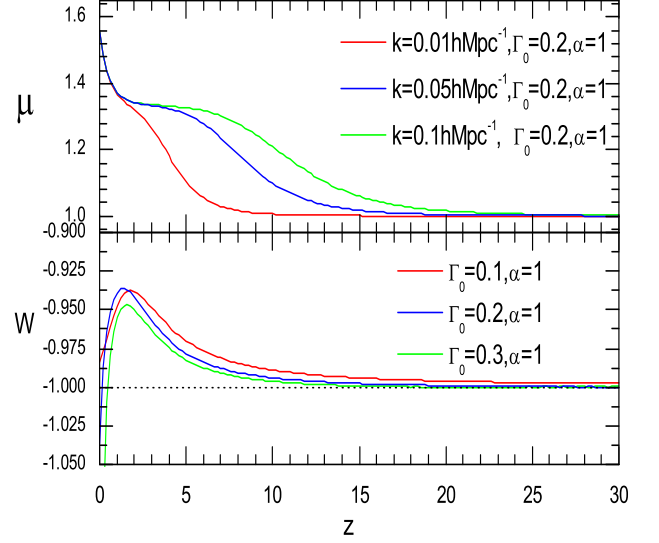$$\frac{dG}{dx} + G^2 + (2 + \frac{d\ln H}{dx})G - \frac{3}{2}\Omega_m\mu(k,x) = 0 \quad , \quad (93)$$

FIG. 2: The evolutions of μ and the effective DE equation of state w in the Jordan frame for the $f(R)$ models we constructed.

where $D(z,k) = \frac{\delta_m(z,k)}{\delta_m(z_i,k)}$ and

$$\mu(k,x) = \frac{1}{F}\frac{4 + 3M^2a(x)^2/k^2}{3(1 + M^2a(x)^2/k^2)} \quad . \qquad (94)$$

The M^2 can be very large in the early time of the universe, since $\Gamma \to 0$ as $R \to \infty$. When $M^2 \gg \frac{a^2}{k^2}$, the regime is called "General Relativistic(GR) regime" where the matter density perturbation has the standard evolution since the modified gravity doesn't show up in this regime where $\delta F \to 0$ and $\Phi/\Psi \to -1$, $\mu(k,x) \to 1$. However, in the late time of the Universe, when $M^2 \ll \frac{a^2}{k^2}$, the modified gravity effect has prominent influence on the structure formation because the ratio of $\Phi/\Psi \to -1/2$, which has been altered greatly from the standard Einstein model and in this case $\mu(k,x) \to \frac{4}{3F_0}$ where F_0 is the value of F today. This regime is called the "scalar-tensor(ST) regime" and obviously matter perturbation does not have the standard evolution as in the Einstein gravity in this regime. As shown in the upper panel of Fig 2, in the transition phase of these two regimes, $\mu(k,x)$ is strongly scale dependent on the wave number of k. The transition of these two regimes can be characterized by $M^2 = k^2/a^2$ which can occur as early as in the matter dominated epoch, even in small scales, in certain types of $f(R)$ models in the expansion history of the Universe [9]. In our model, since the dilation rate function Γ will drop rapidly in the early time of the Universe, the transition usually happens very late $z < 30$ for interested wave number k. Thus in our constructed $f(R)$ models, we have the standard matter dominated phase and matter power spectrum, which can avoid the problems for the usual $f(R)$ model mentioned in [9].

V. CONCLUSIONS

In this work, we have further disclosed the equivalence of the $f(R)$ gravity presented in the Jordan frame and in the Einstein frame from the background dynamics to the perturbation theory. In the Jordan frame we have derived the general analytic solution for the $f(R)$ model to have the same expansion history as the ΛCDM cosmology. This analytic solution of the $f(R)$ model can provide further insights on how cosmology can be used to test the gravity at the largest scales. Moreover we have presented a systematic and self-consistent way to construct the $f(R)$ model in the Jordan frame through conformal transformation by using the mass dilation rate function in the Einstein frame. We have also developed linear perturbation theories to further exhibit the equivalence of the $f(R)$ gravity in the Jordan frame and in the Einstein frame. We have shown that the $f(R)$ model constructed from the mass dilation rate can produce observational allowed effective DE equation of state and provide the matter phase in consistent with cosmological observations. It is of great interest to confront our $f(R)$ model with latest observations in the future study.

It is interesting to note that our $f(R)$ model was constructed from the mass dilation rate, which refers to a fundamental quantity, the mass, rather than the mysterious DE equation of state. This gives the root for understanding the equivalence between the $f(R)$ model in the Jordan frame and the Einstein frame, since we know well that the concept of mass plays a central role in gravitational theories [17, 33]. The mass can be classified into the inertial mass and the gravitational mass. In the Jordan frame the $f(R)$ gravity can be effectively treated with the modified gravitational constant $G_{eff} = \frac{G}{F}$ if we compare it with the Einstein gravity. This is effectively equivalent to rescale the gravitational mass $\tilde{m}_g = m\sqrt{F}$. However the inertial mass in the Jordan frame is conserved so that the equation of motion for a free particle in the Jordan frame is described by $u^\mu \nabla_\mu p^\nu = 0$, which is the same as the description in the standard Einstein gravity, where the rest inertial mass of particle is conserved.

Any change in the gravitational mass in the Jordan frame inevitably changes the gravitational field, which, in turn, will change the inertial frame. Any change in the inertial frame will induce the well-known "frame-dragging" effect which was recently confirmed by the Gravity Probe B mission [34]. The inertial "frame-dragging" effect can be equivalently considered as the "inertial mass-dragging" effect by assuming that the inertial frame is unchanged while the inertial mass of particles defined in this frame is rescaled. This understanding can help us interpret that after the conformal transformation, in the Einstein frame, the equation of motion for particles in the inertial frame is governed by the varying mass equation of motion $\tilde{u}^\mu \tilde{\nabla}_\mu \tilde{p}^\nu = \tilde{m}\tilde{\partial}^\mu \ln \sqrt{F} = \frac{d\tilde{m}}{d\tilde{t}}(\frac{\partial}{\partial \tilde{t}})^\nu$, which is one of the well established physics in General Relativity [26, 35]. The rest inertial mass in the Einstein frame is indeed rescaled as $\tilde{m}_I = m\frac{1}{\sqrt{F}}$ and is no longer conserved.

Considering the Mach principle, there should be no difference among gravitational theories described in different frames. Thus a viable $f(R)$ model in the Jordan frame should be consistent with the viable model in the Einstein frame in describing cosmology. The result obtained in the work has solid physics root.

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